Uncertain destination dynamics

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Certain dynamical systems exhibit a sensitivity to initial conditions in which the asymptotic state is selected from an infinite number of possible states. The phase space of such systems is foliated with ''attractors,'' each of which is associated with a particular set of initial conditions. The associated uncertain destination dynamics can be analyzed by an appropriate reduction of the full system to a subsystem that explicitly yields the dynamics. [S1063-651X(99)09909-2]

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Recent studies of high-dimensional dynamical systems have revealed many new modes of complex evolution, such as synchronization [1], on-off intermittency [2], and fractal, riddled, and intermingled basins [3]. Many of these studies have raised fundamental questions concerning the relationship between model systems and the real systems they attempt to describe [4]. In this paper, we report on another, gross complexity in the evolution of high-dimensional systems.

For many dynamical systems governed by evolution equations of the form

$$\dot{x}_i = f_i(x_1,\ldots,x_n), \quad i=1,\ldots,n \tag{1}$$

the long-time evolution can be described by a reduced system,

$$\dot{x}_i = g_i(x_1,\ldots,x_m;c_1,\ldots,c_k), \quad i=1,\ldots,m, \;\; k \leqslant n-m \tag{2}$$

along with a series of (typically) algebraic relations,

$$0 = h_j(x_1,\ldots,x_n;c_j), \quad j=1,\ldots,k \tag{3}$$

where c_j are constants. In general, these constants will depend on the initial conditions of the full system,

$$c_j = c_j(x_{0i}), \quad i=1,\ldots,n. \tag{4}$$

We may refer to the *destination dynamics* of system (1) as being given by the reduced system (2)–(4). The specific form for (2)–(4) may differ for each ''attractor'' of the original system, but, apart from the initial fast transients, the evolution of the reduced and full systems should be equivalent. In experiment, information on the early, fast evolution may be restricted and the destination dynamics may be all that is available for study.

The evolution of certain dynamical systems, such as those with riddled basins of attraction, may exhibit a complex and uncertain dependence on the initial conditions [5]. In this paper, we investigate a different scenario, one in which the initial conditions determine which asymptotic state is selected from an infinite number of possibilities. We show how the destination dynamics given by a reduced system can be used to understand this behavior. We begin by considering a specific example, comprised of a modified form of the Lorenz equations. We then propose a mechanical model as a possible example of a physical system that would exhibit this behavior.

We consider two Lorenz systems [6] coupled through the variables in the following form:

$$\dot{x}_1 = \sigma(x_2 - x_4), \tag{5a}$$

$$\dot{x}_2 = r x_1 - x_2 - x_1 x_3, \tag{5b}$$

$$\dot{x}_3 = x_1 x_2 - b x_3, \tag{5c}$$

$$\dot{x}_4 = \sigma(x_5 - x_4), \tag{5d}$$

$$\dot{x}_5 = r x_1 - x_5 - x_1 x_6, \tag{5e}$$

$$\dot{x}_6 = x_1 x_5 - b x_6. \tag{5f}$$

The unidirectional coupling of Lorenz systems, with subsystem (x_1-x_3) driving subsystem (x_4-x_6), has been studied by Pecora and Carroll [1], He and Vaidya [7], and Tresser, Worfolk, and Bass [8] in the context of synchronizing chaos. Such coupling has also been used in schemes for communications [9]. Here, we have also allowed the variable x_4 to couple the second subsystem back to the first through Eq. (5a).

Figure 1 shows the behavior of this system for three different initial conditions, with parameter values $\sigma = 10$, $r = 160$, and $b = \frac{8}{3}$. In each case, the system evolves to a different final state, with different periodic solutions in (a) and (b) and an aperiodic response in (c). Other initial conditions produce yet different final states for the same parameter values, and the system may also respond to perturbation with a qualitative change in behavior. This gross sensitivity to initial conditions appears to be different from that found in previous studies, where a system exhibits extreme sensitivity in choosing between a small number of attractors for a given set of parameter values [3].

Figure 2(a) shows the largest Lyapunov exponent of the state resulting from initial conditions corresponding to the indicated values of x_{05} and x_{06}. The initial values of the other variables, as well as the parameters σ, r, and b, were held constant, and the system was allowed to evolve to its

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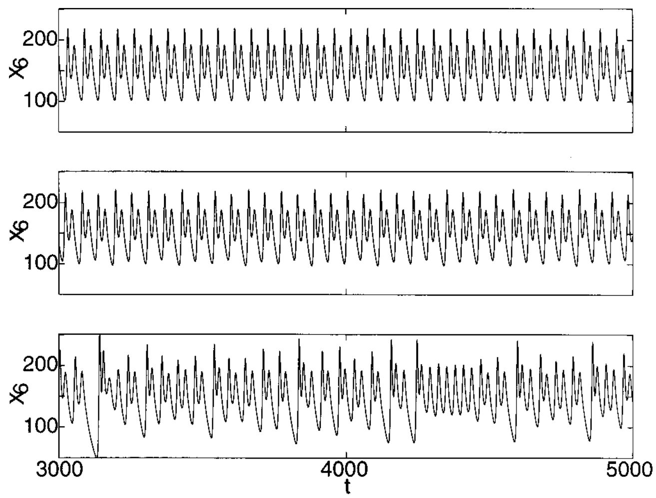

FIG. 1. Time evolution of Eqs. (5) for identical parameter values ($\sigma=10$, $r=160$, and $b=\frac{8}{3}$) but for different initial conditions: (a) period 2, $(x_{05}, x_{06})=(2.0, 0.0)$, (b) period 8, $(x_{05}, x_{06})=(-1.1, 1.0)$, (b) chaos, $(x_{05}, x_{06})=(0.0, -1.0)$. Other initial conditions: $(x_{01}, x_{02}, x_{03}, x_{04})=(0.1, 0.02, 0.02, 1.0)$.

asymptotic state for different x_{05} and x_{06} values. The largest Lyapunov exponent of the corresponding state was then determined and its value was plotted as a function of x_{05} and x_{06} according to the indicated color coding. All periodic states appear as blue, the color assigned to values equal to or less than zero, and chaotic states can be seen as the colors corresponding to positive values. Figure 2(b) shows the maximum amplitude of oscillation $x_6^{\max}$ as a function of the initial value x_{05} for the same initial values of the other variables. The plot corresponds to a one-dimensional (1D) cut at $x_{06}=0$ of the 2D plot in Fig. 2(a), with the color coding again giving the value of the largest Lyapunov exponent.

The dependence on initial conditions shown in Fig. 2 suggests that the six-dimensional phase space is foliated with an infinite number of ''attractors.'' To understand this behavior, we consider a reduced system derived from the full system to determine the destination dynamics. Following the method of He and Vaidya [7], we construct the governing equations for the ''errors'' $e_1=x_1-x_4$, $e_2=x_2-x_5$, and $e_3=x_3-x_6$, giving

$$\dot{e}_1=\sigma e_2, \tag{6a}$$

$$\dot{e}_2=-e_2-x_1 e_3, \tag{6b}$$

$$\dot{e}_3=x_1 e_2-b e_3. \tag{6c}$$

It follows that the function $V=e_2^2+e_3^2$ is a Lyapunov function for this system of equations [10]. Thus e_2 and e_3 must tend to zero, with the pairs of variables (x_2,x_5) and (x_3,x_6) becoming absolutely synchronized, independent of the initial conditions. With $e_2\to 0$, then from Eq. (6a) we also have $e_1=(x_1-x_4)\to c$, where c is some constant dependent on the initial conditions of the full system. The system thus establishes a constant difference between the remaining pair of variables (x_1,x_4). Each new set of initial conditions gives rise to a different value of c for this difference.

The destination dynamics of Eq. (5) can thus be written as

$$\dot{x}_1=\sigma(x_2-x_1+c), \tag{7a}$$

$$\dot{x}_2=r x_1-x_2-x_1 x_3, \tag{7b}$$

$$\dot{x}_3=x_1 x_2-b x_3, \tag{7c}$$

with the constraints

$$x_2-x_5=x_3-x_6=0, \quad x_1-x_4=c. \tag{7d}$$

Compared with the traditional Lorenz system, there is now an additional ''parameter,'' namely, the difference c in Eq. (7a). We can examine the behavior of this system as a function of c, and the corresponding bifurcation sequence is illustrated in Fig. 3. This sequence is symmetric about $c=0$, as Eqs. (7) are invariant under the transformation $(x_1,x_2,c)\to(-x_1,-x_2,-c)$. For $c=0$, the system exhibits a stable period-2 state, but as $|c|$ increases there are various period-doubling and reverse period-doubling cascades with regions of chaos and periodic windows. For the full six-variable system, the value of c corresponding to a given set of initial conditions can be obtained as the long-time limit of the error e_1. For any such value, the destination dynamics of the corresponding three-variable system can also be computed. We find complete correlation between these two approaches, with only a phase shift remaining at long times.

The change in behavior of the full system with the initial conditions can now be understood in terms of the role played by c. The system is comprised of effectively three-dimensional dynamics on invariant manifolds embedded in a four-dimensional space, which is spanned by the ''parameter'' c. For a particular value of c, the evolution on the manifold is governed by the ''attractor'' associated with that value of c. This can be clearly seen by comparing the qualitative features of the bifurcation sequence in Fig. 3 with the initial conditions sequence in Fig. 2(b). Although these sequences do not match quantitatively, since the plot in Fig. 2(b) is a projection of behavior along the ''c direction,'' the qualitative correspondence is striking. Perturbing or changing the initial conditions of the six-variable system corresponds to shifting the dynamics along the transverse c direction and hence along the relatively dense bifurcation sequence.

The behavior described above is not unique to the particular model given by Eqs. (5). We can readily construct other systems of equations exhibiting uncertain destination dynamics by expanding an appropriate set of reduced equations. For example, as a variation on the Lorenz model, starting with the normal three-variable form [Eqs. (7) without the parameter c], we may replace the term $x_1 x_3$ with a new variable x_4, which we require to satisfy the condition $x_4=x_1 x_3+c$, where c is again some constant. For the latter, we require $d(x_4-x_1 x_3)/dt=0$. The resulting four-variable system also exhibits uncertain destination dynamics.

Finally, we present a mechanical system, adapted from that advanced by Sommerer and Ott [5] to illustrate riddled basins of attraction. We consider a unit-mass particle moving in 3D space according to

$$\frac{d^2}{dt^2}\vec{r}=-\gamma\frac{d}{dt}\vec{r}+\vec{F}+\vec{p}\sin\omega t, \tag{8a}$$

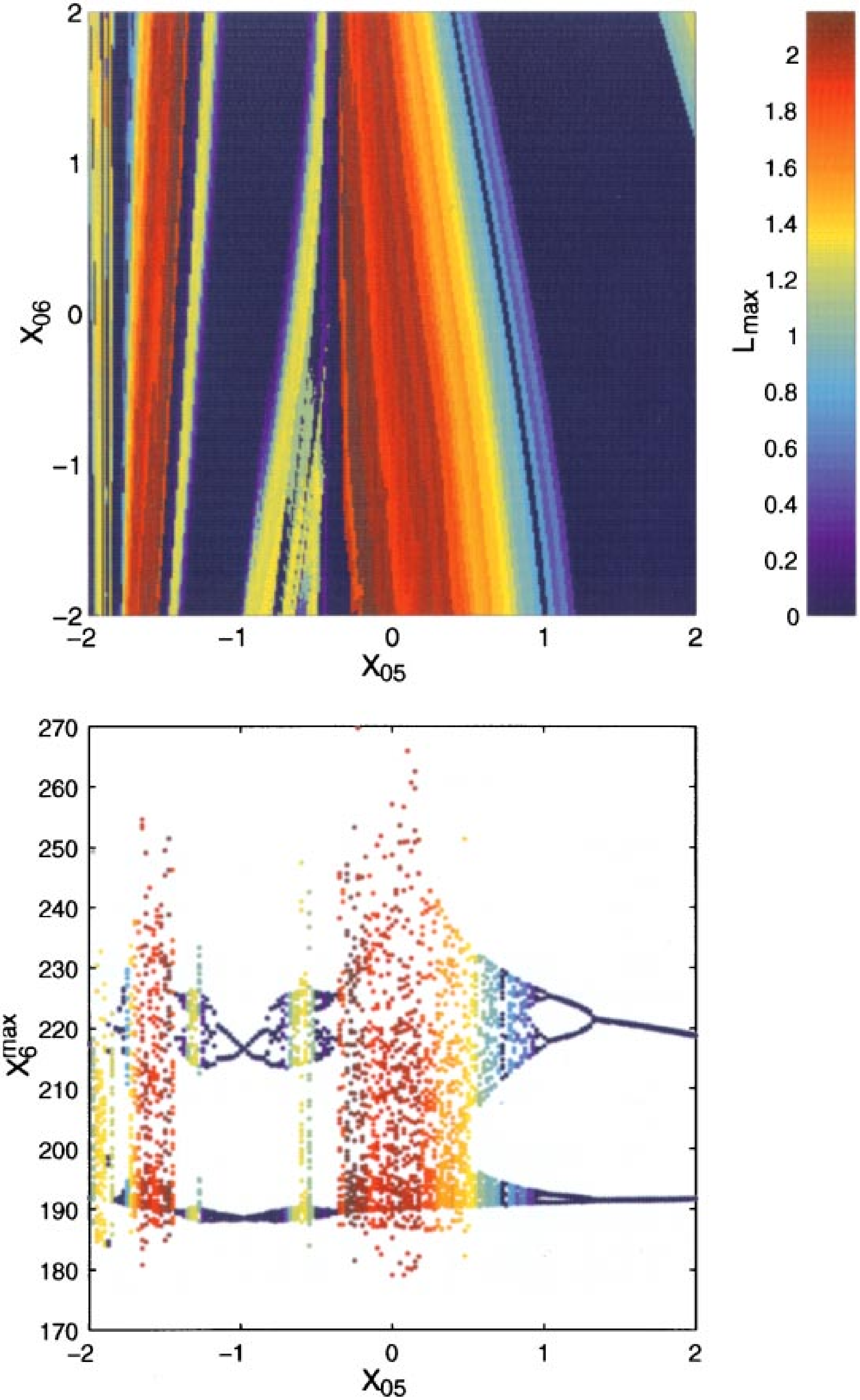

FIG. 2. (Color) (a) Values of the largest Lyapunov exponent corresponding to the state resulting from different initial values x_{05} and x_{06}. See side bar for color coding of exponent values. (b) Values of the amplitude of x_6 as a function of the initial value x_{05} for the initial value $x_{06}=0$. Color coding indicates value of the largest Lyapunov exponent. Parameter values and initial values of other variables as in Fig. 1.

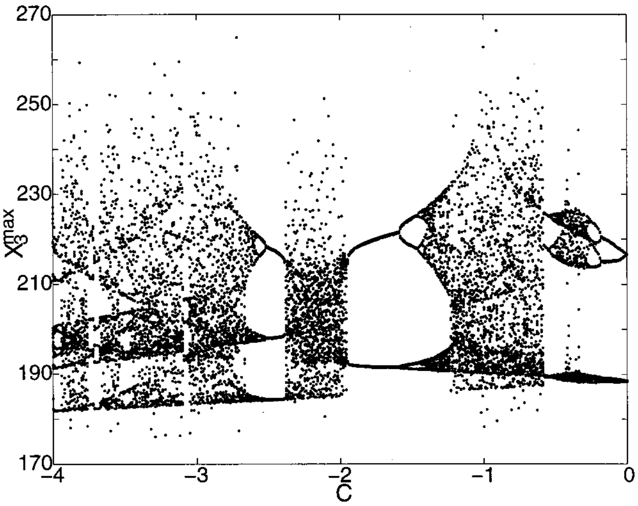

FIG. 3. Bifurcation sequence of destination dynamics given by Eqs. (7) as a function of parameter c. Other parameter values as in Fig. 1.

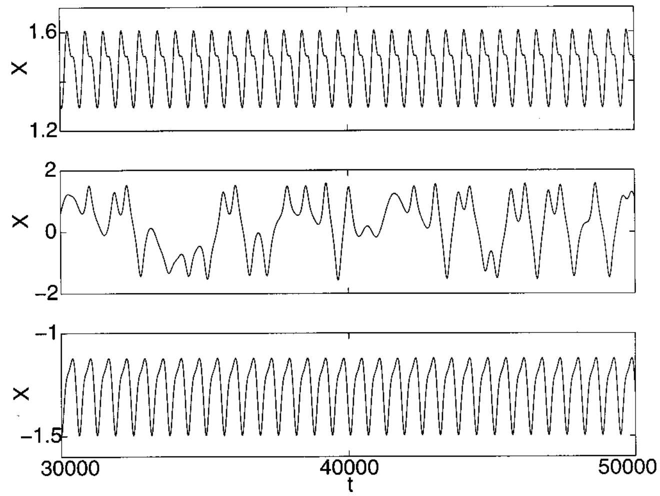

FIG. 4. Time evolution of Eqs. (11) for identical parameter values ($\gamma=0.05$, $p_0=2.3$, $\omega=3.5$) but with different initial conditions: $(x_0, v_{x0}, y_0, v_{y0}, z_0, v_{z0}, t)$ = (0.56, 0.0, 0.06, 0.4, 0.3, 0.01, 0.0), (0.56, 0.0, −0.26, 0.0, 0.003, 0.0, 0.0), (0.56, 0.0, 0.05, 0.0, 0.03, 0.0, 0.0) in (a), (b), and (c).

with $\partial\vec{F}/\partial t=0$, where $\vec{r}=(x,y,z)$ and

$$\vec{F}=-(\nabla V+\nabla\times\vec{A}) \tag{8b}$$

consists of a scalar potential field V and a vortex field $\vec{A}$. Taking, as an example,

$$V=(1-x^2)^2, \quad \vec{A}=\hat{x}x^2z^2+\hat{y}xy^2z+\hat{z}xz^3,$$
$$\text{and with } \vec{p}=p_0\hat{x}, \tag{9}$$

then

$$\vec{F}=[4x(1-x^2)+xy^2]\hat{x}+(z^3-2x^2z)\hat{y}+(-y^2z)\hat{z}. \tag{10}$$

This then gives the equations

$$\frac{d^2x}{dt^2}=-\gamma\frac{dx}{dt}+4x(1-x^2)+xy^2+p_0\sin\omega t, \tag{11a}$$

$$\frac{d^2y}{dt^2}=-\gamma\frac{dy}{dt}+(z^2-2x^2)z, \tag{11b}$$

$$\frac{d^2z}{dt^2}=-\gamma\frac{dz}{dt}-y^2z. \tag{11c}$$

This set of equations can be reduced by noting that $z=0$ is a stable manifold, and with $z\to 0$ we see from Eq. (11b) that $dy/dt\to 0$ and therefore $y\to c$, where c is a constant. This leads to the reduced system

$$\frac{d^2x}{dt^2}=-\gamma\frac{dx}{dt}+4x(1-x^2)+xc^2+p_0\sin\omega t, \tag{12}$$

which is of the form of a forced Duffing equation. This is known to have a dense bifurcation structure for suitable parameter combinations [11], and, as this occurs here with a ''parameter'' c dependent on the initial conditions for the full system, it will show uncertain destination dynamics in a way similar to the coupled Lorenz system, Eqs. (5). Figure 4 shows three examples of the behavior for different initial conditions but with identical parameter values. Again, this result is not specific to the choice of fields, such as Eq. (9) taken here, so we expect the behavior to be generic to a wide class of systems.

In summary, we note that in each example of uncertain destination dynamics, a ''parameter'' c can be identified that attains a particular value following the decay of transient behavior. Different initial conditions or a suitable perturbation give rise to new dynamical behavior corresponding to a different value of c. While the behavior is fully dissipative in the manifold corresponding to a particular value of c, the system has neutral stability to influences that move it transverse to the manifold to a new value of c. It is this neutral stability in the ''c direction'' that gives rise to the foliation of ''attractors'' in phase space and the corresponding dependence on initial conditions.

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